

Spin-wave theory at constant order parameter

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We derive the low-temperature properties of spin- S quantum Heisenberg magnets from the Gibbs free energy $G(M)$ for fixed order parameter M . Assuming that the low-lying elementary excitations of the system are renormalized spin waves, we show that a straightforward $1/S$ expansion of $G(M)$ yields qualitatively correct results for the low-temperature thermodynamics, *even in the absence of long-range magnetic order*. We explicitly calculate the two-loop correction to the susceptibility of the ferromagnetic Heisenberg chain and show that it quantitatively modifies the mean-field result.

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I. INTRODUCTION

For many years the magnetically ordered state of quantum Heisenberg magnets has been studied with the help of the spin-wave expansion.¹ This expansion is usually implemented by expressing the components of the spin operator $\hat{\mathbf{S}}_i = (\hat{S}_i^x, \hat{S}_i^y, \hat{S}_i^z)$ at lattice site i in terms of canonical boson operators \hat{b}_i and \hat{b}_i^\dagger , using either the Holstein-Primakoff transformation² or the Dyson-Maleev transformation.^{3,4} For example, the Dyson-Maleev transformation for a spin- S ferromagnet is

$$\hat{S}_i^+ = (2S)^{1/2} \left[1 - \hat{b}_i^\dagger \hat{b}_i / 2S \right] \hat{b}_i, \quad (1a)$$

$$\hat{S}_i^- = (2S)^{1/2} \hat{b}_i^\dagger, \quad (1b)$$

$$\hat{S}_i^z = S - \hat{b}_i^\dagger \hat{b}_i, \quad (1c)$$

where $\hat{S}_i^\pm = \hat{S}_i^x \pm i\hat{S}_i^y$, and the number of bosons at a lattice site may not exceed $2S$ in order to faithfully represent the $2S+1$ eigenstates of \hat{S}_i^z . If long-range magnetic order is present in the z direction and the number of bosons is small compared to $2S$, the two-body bosonic interaction arising from the term in Eq. (1a) involving three bosonic operators can be treated within conventional many-body perturbation theory, controlled by the formal small parameter $1/S$. Due to the maximum-occupancy constraint the effective bosonic Hamiltonian implicitly contains projection operators that eliminate the unphysical part of the bosonic Hilbert space. Usually, these projection operators are simply ignored. For Heisenberg ferromagnets in three dimensions it was shown by Dyson³ that in the thermodynamic limit the low-temperature thermodynamics can indeed be obtained without taking into account the so-called kinematical interactions associated with these projection operators.

While in the 1960s and 1970s ordered magnets have been intensely studied, in recent years the center of attention has shifted to low-dimensional magnets without broken symmetries. In this case the conventional spin-wave approach described above is not applicable, because it relies on the existence of long-range magnetic order. Nevertheless, in many magnetic materials the elementary excitations still resemble the spin waves of an or-

dered magnet. For example, in two-dimensional quantum Heisenberg ferromagnets⁵ and antiferromagnets⁶ at low but finite temperatures, where the order parameter correlation length ξ is exponentially large, spin waves with wave vectors $|\mathbf{k}| \gg \xi^{-1}$ are well-defined elementary excitations.⁷ Other examples for systems where the low-energy physics is dominated by elementary excitations of the spin-wave type are Haldane-gap antiferromagnets (i.e. one-dimensional Heisenberg antiferromagnets with integer spin S) and one-dimensional Heisenberg ferromagnets with arbitrary spin.

To study the low-temperature properties of these systems, several methods have been proposed. The Schwinger-boson mean-field theory of Arovas and Auerbach⁸ is perhaps aesthetically most appealing. However, going beyond the mean-field approximation within the Schwinger-boson approach has turned out to be quite difficult.⁹ At the mean-field level the modified spin-wave theory (MSWT) proposed by Takahashi¹⁰ is an alternative to the Schwinger-boson approach. MSWT yields results that agree with the predictions of Schwinger-boson mean-field theory up to numerical prefactors. This is not surprising, because both approaches are in fact equivalent to a one-loop renormalization group calculation.^{5,6} Recently, Takahashi's MSWT has also been applied to more complex problems, such as frustrated¹¹ or disordered magnets,¹² or magnetic molecular clusters.¹³ However, the MSWT has shortcomings: (i) it is very difficult to systematically calculate corrections due to interactions between spin waves within MSWT and (ii) the absence of long-range magnetic order is not obtained as a result, i.e., the magnetization is set to zero by hand; this leads to ambiguity in the choice of the constraint if the MSWT is applied to systems with more complicated magnetic order, such as ferrimagnets.¹⁴ In this work we shall show that these problems can be resolved within the conventional spin-wave approach simply by performing the calculation at *constant order parameter*.

This paper is organized as follows. In Sec. II we discuss the calculation of thermodynamic observables at constant order parameter. In Sec. III this approach is applied to the Heisenberg ferromagnet in $D = 1, 2, 3$ dimensions within linear spin-wave theory. Hartree-Fock and

two-loop corrections are obtained for the one-dimensional case in Sec. IV. The work is summarized in Sec. V.

II. THERMODYNAMICS AT CONSTANT ORDER PARAMETER

In this section we discuss the calculation of thermodynamic observables at constant order parameter. Although this approach is applicable to a variety of correlated systems with order parameter \hat{M} and corresponding conjugate field h , here we will focus on the spin- S Heisenberg ferromagnet with zero-field Hamiltonian

$$\hat{H} = -J \sum_{\langle ij \rangle} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j, \quad (2)$$

where the sum is over all nearest-neighbor pairs of a D -dimensional hypercubic lattice with N sites, and $J > 0$ is the exchange coupling. In this case the order parameter is simply the total magnetization, given by $\hat{M} = \sum_{i=1}^N \hat{S}_i^z$, and h is the homogeneous magnetic field (in suitable units). Applications to antiferromagnets or more complicated magnetic systems are straightforward.

Let us first recall some elementary thermodynamics. For fixed field h and temperature T , thermodynamic observables can be obtained from the Helmholtz free energy (setting the Boltzmann constant to unity)

$$F(h) = -T \ln \text{Tr} e^{-(\hat{H} - h\hat{M})/T}, \quad (3)$$

where the dependence on T is suppressed for brevity. Given $F(h)$, the magnetization is obtained as

$$M(h) = -\frac{\partial F(h)}{\partial h}. \quad (4)$$

Alternatively, we may choose to fix the magnetization and adjust the magnetic field appropriately. The corresponding thermodynamic potential is the Gibbs free energy $G(M)$, which is related to the Helmholtz free energy via a Legendre transformation,¹⁵

$$\begin{aligned} G(M) &= h(M)M + F(h(M)) \\ &= -T \ln \text{Tr} e^{-[\hat{H} - h(M)(\hat{M} - M)]/T}, \end{aligned} \quad (5)$$

where the function $h(M)$ is obtained from Eq. (4). From $G(M)$ we obtain the equation of state in the form $h = h(M)$ via

$$h(M) = \frac{\partial G(M)}{\partial M}, \quad (6)$$

which shows that the equilibrium magnetization for vanishing field is an extremum of $G(M)$. If the system has a finite spontaneous magnetization $M_0 = \lim_{h \rightarrow 0^+} M(h)$, then the generic expected behavior of $G(M)$ is, for $M \geq M_0$,

$$G(M) = G(M_0) + \frac{(M - M_0)^2}{2\chi} + O[(M - M_0)^3], \quad (7)$$

while for $|M| < M_0$ the Gibbs free energy has the constant value $G(M_0)$; see, for example, Ref. 16. Here

$$\chi^{-1} = \left. \frac{\partial h(M)}{\partial M} \right|_{M_0} = \left. \frac{\partial^2 G(M)}{\partial M^2} \right|_{M_0} \quad (8)$$

is the inverse longitudinal order parameter susceptibility for vanishing external field. These expressions are also valid in the absence of spontaneous symmetry breaking, where $M_0 = 0$. Note that, in general, $G(M) = G(-M)$, because the spectrum of \hat{M} is symmetric with respect to the origin.

The parameter $h(M)$ in Eq. (5) can be viewed as a Lagrange multiplier that enforces the condition of constant magnetization. The zero-temperature version of the method outlined above has been used previously by Georges and Yedidia¹⁷ to study spontaneous symmetry breaking in the ground state of the Hubbard model. Note that in the limit $T \rightarrow 0$ Eq. (5) can be written as¹⁸ $G(M) = \langle 0 | \hat{G}(M) | 0 \rangle$, where $|0\rangle$ is the ground state of the “free-energy operator” $\hat{G}(M) = \hat{H} - h(M)[\hat{M} - M]$. As shown in Ref. 17, the expansion at constant order parameter is advantageous for the calculation of corrections to the mean-field approximation. In the following section we show that for low-dimensional Heisenberg magnets without long-range order this method yields reasonable results even at the level of linear spin-wave theory; the leading fluctuation corrections in $D = 1$ are then calculated in Sec. IV.

III. LINEAR SPIN WAVES AT CONSTANT ORDER PARAMETER

We now calculate $G(M)$ within linear spin-wave theory, i.e., to leading order in $1/S$, assuming that the low-lying elementary excitations of the system are renormalized spin waves. In this approximation the square brackets in Eq. (1a) are simply replaced by unity, so that the Heisenberg Hamiltonian (2) becomes

$$\hat{H}_0 = -DJNS^2 + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}}, \quad (9)$$

where $\epsilon_{\mathbf{k}} = 2DJS(1 - \gamma_{\mathbf{k}})$, with

$$\gamma_{\mathbf{k}} = D^{-1} \sum_{\mu=1}^D \cos(\mathbf{k} \cdot \mathbf{a}_\mu). \quad (10)$$

For simplicity, we impose periodic boundary conditions on a hypercubic lattice with primitive lattice vectors \mathbf{a}_μ and lattice spacing $a = |\mathbf{a}_\mu|$. The momentum sum is over the first Brillouin zone and $\hat{b}_{\mathbf{k}}$ is the lattice Fourier transform of \hat{b}_i . The corresponding free energy is

$$F_0(h) = -DJNS^2 - hNS + T \sum_{\mathbf{k}} \ln \left[1 - e^{-(\epsilon_{\mathbf{k}} + h)/T} \right]. \quad (11)$$

From Eq. (4) we then obtain the usual spin-wave result for the magnetization

$$M(h) = NS - \sum_{\mathbf{k}} [e^{(\epsilon_{\mathbf{k}}+h)/T} - 1]^{-1}. \quad (12)$$

A. Three-dimensional ferromagnet

It is instructive to begin with linear spin-wave theory for the three-dimensional Heisenberg model. In the thermodynamic limit we obtain for the magnetization per site $m = M/N$ to leading order in $t = T/JS$ and $v = h/T$

$$m(h) = S - \frac{\zeta(\frac{3}{2})}{8\pi^{3/2}} t^{3/2} + \frac{1}{4\pi} t^{3/2} v^{1/2} + O(t^{5/2}, t^{3/2} v), \quad (13)$$

where $\zeta(z)$ is the zeta function. Setting $h = 0$ we recover the well-known Bloch $T^{3/2}$ law for the spontaneous magnetization per site, $m_0 = \lim_{h \rightarrow 0+} m(h)$, in the ordered state of the Heisenberg ferromagnet. Taking the derivative of Eq. (13) with respect to h , we see that the susceptibility $\chi = \partial M / \partial h$ [Eq. (8)] diverges for $h \rightarrow 0$ as $h^{-1/2}$. This divergence of the uniform longitudinal susceptibility of a three-dimensional Heisenberg magnet in the ordered state is not widely appreciated, although it was noticed a long time ago^{2,3} and has been confirmed by renormalization group calculations for the classical Heisenberg ferromagnet^{19,20} and perturbative calculations for the corresponding quantum model.^{21,22} Due to this divergence, the Gibbs free energy $G(M)$ of the Heisenberg ferromagnet in $D = 3$ does not have the generic form (7). Instead the linear spin-wave result for $G(M)$ is, for $m \geq m_0$,

$$\frac{G_0(M)}{NT} = -\frac{3JS^2}{T} - \frac{\zeta(\frac{5}{2})}{8\pi^{3/2}} t^{3/2} + \frac{16\pi^2}{3t^3} (m - m_0)^3 + O[(m - m_0)^4]. \quad (14)$$

From Eq. (13) we note that $h^{1/2} \propto (m - m_0)$, so that we cannot solve for h as a function of m unless $m > m_0$. In light of the above general discussion this is not surprising, because for $|m| < m_0$ the Gibbs free energy is constant.¹⁶ The behavior of $G_0(M)$ as a function of M is shown in Fig. 1. The leading m dependence of Eq. (14) is proportional to $(m - m_0)^3$, which can be traced to the fact that the inverse susceptibility vanishes. By contrast, in $D > 4$ the uniform longitudinal susceptibility of the Heisenberg ferromagnet is finite,^{19,20} so that in this case the Gibbs free energy has indeed the generic form (7).

B. One-dimensional ferromagnet

Let us consider now the one-dimensional case, where we know that the Heisenberg ferromagnet does not have any long-range order at any finite temperature T . In this

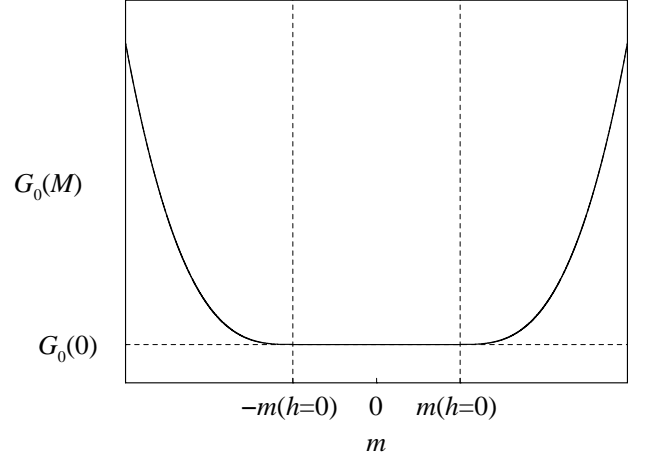


FIG. 1: Gibbs free energy $G_0(M)$ of the three-dimensional Heisenberg ferromagnet within linear spin-wave theory. Due to the divergent longitudinal susceptibility in $D = 3$, the Gibbs free energy grows cubically for $|m|$ slightly above m_0 , see Eq. (14).

case the linear spin-wave theory result for the magnetization per site is

$$\frac{m(h)}{S} = 1 - \frac{\zeta(\frac{1}{2})}{2S\sqrt{\pi}} \sqrt{t} - \frac{1}{2S} \sqrt{\frac{t}{v}} + O(t, t^{3/2} v^{-1/2}), \quad (15)$$

where again $t = T/JS$ and $v = h/T$. This expression predicts a divergent magnetization and susceptibility for $h \rightarrow 0$. However, we can obtain a perfectly finite result for the susceptibility at constant magnetization [Eq. (8)]. Solving Eq. (15) for h as a function of $M = Nm$ we obtain

$$h(M) = \frac{T^2}{4JS[S - m - \frac{\zeta(\frac{1}{2})}{2\sqrt{\pi}} \sqrt{t}]^2}. \quad (16)$$

According to Eq. (8) this implies for the inverse susceptibility

$$\chi^{-1} = \frac{T^2}{2NJS[S - m - \frac{\zeta(\frac{1}{2})}{2\sqrt{\pi}} \sqrt{t}]^3}. \quad (17)$$

Anticipating that in one dimension $m = 0$, we obtain for the susceptibility per site at low temperatures

$$\frac{\chi}{N} = \frac{2JS^4}{T^2} \left[1 - \frac{3}{S} \frac{\zeta(\frac{1}{2})}{2\sqrt{\pi}} \sqrt{t} + O(t) \right]. \quad (18)$$

This expression agrees exactly²³ with the prediction of the MSWT advanced by Takahashi,¹⁰ who argued that Eq. (18) is indeed the correct asymptotic low-temperature behavior of the susceptibility for arbitrary S . For $S = 1/2$ the nearest-neighbor Heisenberg chain is exactly solvable via Bethe ansatz,²⁴ so that in this case one can obtain an independent check of Eq. (18). Indeed, from a numerical analysis of the Bethe-ansatz integral equations²⁴ Takahashi¹⁰ found perfect agreement

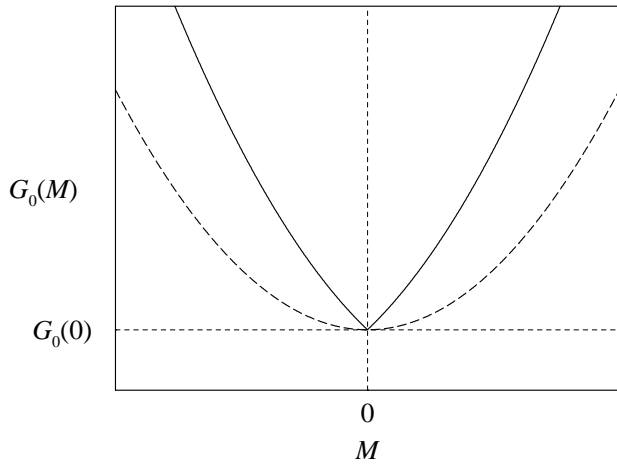


FIG. 2: Solid line: Gibbs free energy $G_0(M)$ of the one-dimensional Heisenberg ferromagnet within linear spin-wave theory, see Eq. (19). The cusp at $m = 0$ is an unphysical artefact of the spin-wave expansion, which is related to the neglect of the kinematical interaction between the spin waves. The dashed line is the subtracted Gibbs free energy $\tilde{G}_0(M) = G_0(M) - h(0)|M|$.

with Eq. (18) for $S = 1/2$, which is remarkable because a priori linear spin-wave theory is only expected to be accurate in the ordered state and for large S . We shall further comment on this agreement below.

Within linear spin-wave theory, the Gibbs free energy per site is given by

$$\frac{G_0(M)}{NT} = -\frac{JS^2}{T} - \frac{\zeta(\frac{3}{2})}{2\sqrt{\pi}}\sqrt{t} + \frac{1}{NT} \left[h(0)|M| + \frac{M^2}{2\chi} + O(|M|^3) \right], \quad (19)$$

where χ is given in Eq. (18) and

$$h(0) = \frac{T^2}{4JS^3} + O(T^{5/2}), \quad (20)$$

see Eq. (16). In writing Eq. (19) we have used the fact that our spin-wave calculation yields $G_0(M)$ only for $M \geq 0$ and that the exact $G(M)$ is an even function of M . Note that $G_0(M)$ assumes a minimum at $M = 0$, indicating the absence of long-range order. However, as shown in Fig. 2, linear spin-wave theory predicts an unphysical cusp in the Gibbs free energy at $M = 0$. The finite slope $h(0) = \partial G_0 / \partial M|_{M=0+}$ can be identified with the variational parameter $-\mu$ introduced by Takahashi,¹⁰ which in his calculation plays the role of a chemical potential for the Dyson-Maleev bosons, enforcing the condition of zero magnetization. On the other hand, it is physically clear that for $T > 0$ any finite value of the external field will always be accompanied with a finite magnetization, so that an exact calculation of $G(M)$ should yield $\lim_{M \rightarrow 0} h(M) = 0$. Therefore we expect that the exact Gibbs free energy in one dimension has the form

given in Eq. (7). The cusp of the spin-wave result for the Gibbs free energy is related to the fact that in our simple spin-wave calculation we have ignored the *kinematical* interaction between the spin waves which arises from the maximum-occupancy constraint.²⁵ Fortunately, this cusp is irrelevant for the calculation of the zero-field thermodynamics, which can be also obtained from the subtracted Gibbs free energy

$$\tilde{G}_0(M) = G_0(M) - h(0)|M|, \quad (21)$$

see Fig. 2. Note that $\tilde{G}_0(M)$ has the generic behavior given in Eq. (7), with the susceptibility given by Eq. (18).

C. Two-dimensional ferromagnet

For completeness we now discuss the case $D = 2$, where the spontaneous magnetization of the Heisenberg ferromagnet is zero at any finite temperature T . The result of linear spin-wave theory for the magnetization is (with $t = T/JS$, $v = h/T$)

$$m(h) = S - \frac{t}{4\pi} [-\ln v + \frac{v}{2} + \frac{\zeta(2)}{8}t + O(t^2, v^2)], \quad (22)$$

which again diverges for $h \rightarrow 0$. The function $h(m)$ is obtained as

$$h(m) = T e^{4\pi(S-m)/t} [1 + O(t)], \quad (23)$$

and the result for the susceptibility at $m = 0$ is

$$\chi = \frac{e^{4\pi S/t}}{4\pi JS} [1 + O(t)], \quad (24)$$

which diverges for $T \rightarrow 0$. These expressions are analogous to Takahashi's results.¹⁰ Finally, the Gibbs free energy takes the form

$$\frac{G_0(M)}{NT} = -\frac{\zeta(2)}{4\pi}t + e^{-4\pi S/t} \left(\frac{t}{4\pi}|m| + \frac{m^2}{2} \right) + O(t, m^3), \quad (25)$$

i.e., with a minimum at $m = 0$, again with an unphysical cusp; in this sense the situation is rather similar to that in $D = 1$. Note, however, that in $D = 2$ it is known that a two-loop calculation is necessary to obtain the correct low-temperature asymptotics of the susceptibility.⁵ Although the exponential factor $\chi \propto \exp[4\pi JS^2/T]$ is correctly reproduced by mean-field theory, the two-loop correction changes the power of T in the prefactor of Eq. (24); the correct low-temperature behavior of the susceptibility of the quantum Heisenberg ferromagnet in two dimensions is $\chi \propto T^2 \exp[4\pi JS^2/T]$. This result is not modified if higher-order terms involving more than two loops are included.^{5,6}

IV. BEYOND LINEAR SPIN-WAVE THEORY

Because fluctuation effects are usually stronger in lower dimensions, we expect that in one dimension the corrections to the mean-field result (18) are even more important than in $D = 2$. We now explicitly calculate the two-loop correction. Within the Dyson-Maleev formalism the dynamical spin-wave interactions are contained in the following two-body Hamiltonian:

$$\hat{H}_1 = \frac{DJ}{N} \sum_{\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}_2, \mathbf{k}_1} \delta_K(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_2 + \mathbf{k}_1) \times V(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}_2, \mathbf{k}_1) \hat{b}_{\mathbf{k}'_1}^\dagger \hat{b}_{\mathbf{k}'_2}^\dagger \hat{b}_{\mathbf{k}_2} \hat{b}_{\mathbf{k}_1}, \quad (26)$$

where $\delta_K(\mathbf{k})$ denotes momentum conservation modulo a reciprocal-lattice vector, and the symmetrized interaction vertex is

$$V(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}_2, \mathbf{k}_1) = -\frac{1}{4} \left[\gamma_{\mathbf{k}_1 - \mathbf{k}'_1} + \gamma_{\mathbf{k}_1 - \mathbf{k}'_2} + \gamma_{\mathbf{k}_2 - \mathbf{k}'_1} + \gamma_{\mathbf{k}_2 - \mathbf{k}'_2} - 2\gamma_{\mathbf{k}'_1} - 2\gamma_{\mathbf{k}'_2} \right], \quad (27)$$

with $\gamma_{\mathbf{k}}$ defined in Eq. (10). First let us estimate the effect of \hat{H}_1 within the self-consistent Hartree-Fock approximation. We write our spin-wave Hamiltonian as $(\hat{H}_0 + \delta\hat{H}_1) + (\hat{H}_1 - \delta\hat{H}_1)$ and choose the one-body Hamiltonian $\delta\hat{H}_1$ such that the thermal expectation value of the residual interaction $\hat{H}_1 - \delta\hat{H}_1$ in the ensemble defined by the Hartree-Fock Hamiltonian $\hat{H}_0 + \delta\hat{H}_1$ vanishes. We obtain

$$\delta\hat{H}_1 = \sum_{\mathbf{k}} \Sigma_1(\mathbf{k}) \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} - \frac{2DJ}{N} \sum_{\mathbf{k}, \mathbf{k}'} V(\mathbf{k}, \mathbf{k}', \mathbf{k}', \mathbf{k}) n_{\mathbf{k}} n_{\mathbf{k}'}, \quad (28)$$

where $n_{\mathbf{k}} = [e^{(E_{\mathbf{k}} + h)/T} - 1]^{-1}$ is the thermal occupation of the Hartree-Fock magnon states with momentum \mathbf{k} , and the Hartree-Fock self-energy is given by

$$\Sigma_1(\mathbf{k}) = \frac{4DJ}{N} \sum_{\mathbf{k}'} V(\mathbf{k}, \mathbf{k}', \mathbf{k}', \mathbf{k}) n_{\mathbf{k}'}. \quad (29)$$

After some standard manipulations we obtain for the Helmholtz free energy within self-consistent Hartree-Fock approximation

$$F_1(h) = -DJNS^2 - hNS + T \sum_{\mathbf{k}} \ln \left[1 - e^{-(E_{\mathbf{k}} + h)/T} \right] + DJNS^2(1 - Z)^2. \quad (30)$$

Here $E_{\mathbf{k}} = Z\epsilon_{\mathbf{k}}$, and the dimensionless renormalization factor Z satisfies the self-consistency condition

$$Z = 1 - \frac{1}{NS} \sum_{\mathbf{k}} (1 - \gamma_{\mathbf{k}}) n_{\mathbf{k}}. \quad (31)$$

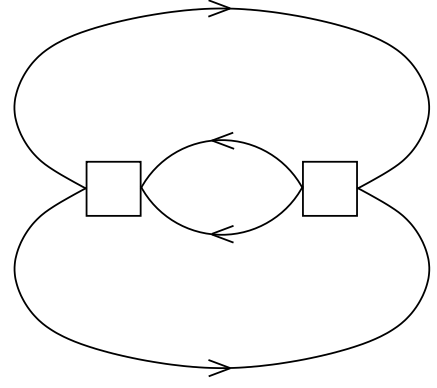


FIG. 3: Feynman diagram describing the leading fluctuation correction to the free energy of the ferromagnetic Heisenberg model, see Eq. (33). The solid arrows denote the Hartree-Fock magnon propagators and the squares are the Dyson-Maleev vertices.

The quantity ZS corresponds to the second variational parameter S' introduced by Takahashi.¹⁰ Note that he gives a different sign for the last term in Eq. (30). In one dimension $1 - Z = O(T^2)$ at low temperatures,¹⁰ so that for the calculation of the first two terms in low-temperature expansion of thermodynamic observables it is sufficient to set $Z = 1$. We conclude that at the Hartree-Fock level the dynamical interaction between spin waves does not contribute to the low-temperature asymptotics in $D = 1$. At this level of approximation our theory is equivalent to MSWT.

Within our approach it is now straightforward to study spin-wave interactions beyond the Hartree-Fock approximation. Therefore we simply expand the Helmholtz free energy $F(h)$ to higher order in the interaction and then perform a Legendre transformation to obtain the corresponding Gibbs free energy. We now calculate the first fluctuation correction to $F(h)$. The relevant Feynman diagram is shown in Fig. 3. In this approximation the Helmholtz free energy is $F_2(h) = F_1(h) + \delta F_2(h)$, where $F_1(h)$ is given in Eq. (30) and

$$\delta F_2(h) = 2 \left(\frac{DJ}{N} \right)^2 \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}} \frac{W(\mathbf{k}, \mathbf{k}', \mathbf{q})}{E_{\mathbf{k}} + E_{\mathbf{k}'} - E_{\mathbf{k} + \mathbf{q}} - E_{\mathbf{k}' - \mathbf{q}}} \times \left[(1 + n_{\mathbf{k}})(1 + n_{\mathbf{k}'})n_{\mathbf{k} + \mathbf{q}}n_{\mathbf{k}' - \mathbf{q}} - n_{\mathbf{k}}n_{\mathbf{k}'}(1 + n_{\mathbf{k} + \mathbf{q}})(1 + n_{\mathbf{k}' - \mathbf{q}}) \right], \quad (32)$$

where

$$W(\mathbf{k}, \mathbf{k}', \mathbf{q}) = V(\mathbf{k}, \mathbf{k}', \mathbf{k} + \mathbf{q}, \mathbf{k}' - \mathbf{q}) \times V(\mathbf{k} + \mathbf{q}, \mathbf{k}' - \mathbf{q}, \mathbf{k}, \mathbf{k}'). \quad (33)$$

At low temperatures, we may replace the Dyson-Maleev vertex by its long-wavelength limit, which in D dimensions is given by $V(\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}_2, \mathbf{k}_1) \sim -(\mathbf{k}_1 \cdot \mathbf{k}_2)a^2/2D$. For the rest of this work we shall explicitly set $D = 1$. Then the leading behavior of $\delta F_2(h)$ for small $t = T/JN$ and

small $v = \hbar/T$ can be calculated analytically. We obtain for $N \rightarrow \infty$

$$\frac{\delta F_2(h)}{TN} = \frac{1}{16} \frac{t^{3/2}}{(2S)^2 v^{1/2}} + O(t^{3/2}, t^{1/2} v). \quad (34)$$

The resulting equation of state is

$$\begin{aligned} \frac{m(h)}{S} = 1 - \frac{\zeta(\frac{1}{2})}{2S\sqrt{\pi}} \sqrt{t} - \frac{1}{2S} \sqrt{\frac{t}{v}} \\ + \frac{1}{16} \left[\frac{1}{2S} \sqrt{\frac{t}{v}} \right]^3 + O(t, t^{3/2} v^{-1/2}). \end{aligned} \quad (35)$$

Comparing this result with the corresponding expression obtained within linear spin-wave theory given in Eq. (15), we see that the two-loop correction gives rise to an additional term proportional to the third power of $(2S)^{-1}(t/v)^{1/2}$. But linear spin-wave theory predicts that this parameter is actually close to unity, as is easily seen by setting $m = 0$ in Eq. (15). Hence, *the leading fluctuation correction to the Hartree-Fock theory is not controlled by a small parameter*. Note that the extra power of S^{-1} that appears in the two-body part of the effective boson Hamiltonian is canceled by the singular \hbar dependence of the two-loop correction. If we nevertheless truncate the expansion at the two-loop order, we obtain from Eq. (35) for the leading low-temperature behavior of the susceptibility,

$$\frac{\chi}{N} \sim C_\chi \frac{JS^4}{T^2}, \quad (36)$$

with $C_\chi \approx 1.96$, which is slightly smaller than the linear spin-wave prediction $C_\chi = 2$, and significantly smaller than the result $C_\chi = 3$ obtained within Schwinger-boson mean-field theory.⁸ We suspect that corrections involving more loops will involve higher powers of the parameter $(2S)^{-1}(t/v)^{1/2}$ in Eq. (36), which give rise to additional finite renormalizations of C_χ . Hence, a numerically accurate expression for the low-temperature susceptibility of a one-dimensional Heisenberg ferromagnet cannot be obtained from a truncation of the $1/S$ spin-wave expansion at some finite order. Note that quantum Monte Carlo simulations for the $S = 1/2$ nearest-neighbor Heisenberg chain²⁶ give $C_\chi = 1.58 \pm 0.03$, supporting the scenario described above. In light of these results it is puzzling that from the numerical analysis of the Bethe-ansatz integral equations for $S = 1/2$ Takahashi^{10,24} obtained $C_\chi = 2$. Possibly this is related to difficulties in extracting the true asymptotic low-temperature behavior of the susceptibility from the Bethe-ansatz integral equations.^{10,27}

Although at first sight our spin-wave expansion for the susceptibility appears to be controlled by the small parameter $1/S$, this parameter is renormalized by the infrared singularity of the two-loop correction to the mean-field result. This phenomenon is familiar from the weak-coupling calculation of the two-loop correction to the ground-state energy of the repulsive Hubbard model at constant staggered magnetization in one and two dimensions.^{17,28} Due to infrared singularities inherent in the loop integrals, those expansions are effectively in powers of the Hubbard interaction U multiplied by a function of the order parameter; as a consequence the two-loop correction to the ground-state energy has the same order of magnitude as the Hartree-Fock term.

V. CONCLUSIONS

In summary, we have shown that conventional spin-wave expansion at constant order parameter is an alternative to Takahashi's modified spin-wave theory, which nowadays is one of the most popular mean-field methods to study low-dimensional magnets without long-range order. Due to the conceptual simplicity of our method, we can systematically calculate corrections to the mean-field approximation using conventional diagrammatic methods. We have explicitly calculated the leading fluctuation correction to the mean-field result for the susceptibility of the ferromagnetic Heisenberg chain. We have found that in one dimension the predictions of MSWT are at most qualitatively correct, because fluctuation corrections are not controlled by a small parameter.

Furthermore, compared to MSWT the present approach has the conceptual advantage that the introduction of Lagrange multipliers by hand is not necessary, since their role is played by external fields instead. The difference between these approaches is best visible for systems with more complicated order parameters. For example, recent work on molecular magnets¹³ and ferrimagnets¹⁴ has shown that the formulation of MSWT for such systems is difficult and that the proper choice of constraints is not clear a priori. On the other hand, spin-wave theory at constant order parameter naturally yields the absence of long-range order in low dimensions *as a result*; applications to antiferromagnets and ferrimagnets, with vanishing homogeneous and staggered magnetizations, are in progress.²⁵

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